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LETTER TO THE EDITOR

Energy dissipation and Kolmogorov law in turbulent flows

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Abstract. We show that energy dissipation is Reynolds number independent in three-dimensional turbulent flows. We obtain the small-scale Kolmogorov spectrum without any particular assumption on energy dissipation, using an approach developed in a previous paper.

In 1941 Kolmogorov formulated his theory on the small-scale dynamics of turbulent flows using two principal assumptions (Monin and Yaglom 1975):

(a) energy transport among all the scales of a turbulent flow is local in phase space (in other words small eddies are only advected by large eddies);

(b) far from the dissipation range, the average energy dissipation $\langle\epsilon\rangle$ and the scale of motion are the only parameters describing the statistical properties at that scale.

The assumption (b) is strongly based on the experimental fact (Kuo and Corrsin 1971) that $\langle\epsilon\rangle$ is Reynolds number independent for sufficiently high Reynolds numbers.

From assumptions (a) and (b), and neglecting fluctuations in energy dissipation, it is easy to show by dimensional consideration that the energy spectrum $E(K)$ of a turbulent flow is in the inertial range

$$E(K)=C\langle\epsilon\rangle^{2/3}K^{-5/3} \tag{1}$$

where C is a universal quantity. $\int_0^\infty E(K)\,\mathrm{d}K$ is the kinetic energy of the flow per unit volume. We can think of $E(K)$ as the average kinetic energy at scale $l=1/K$.

In a recent work two of the present authors (Benzi and Vulpiani 1980, hereafter referred to as I), using a simple and new idea on the dynamics of a turbulent flow, have computed the correction to the Kolmogorov law due to the intermittence phenomenon. In I, we used both assumptions (a) and (b), i.e. we estimated the correction to equation (1) by computing the statistical properties of energy dissipation and using the Kolmogorov theory as the zeroth approximation. In this Letter we shall investigate whether assumption (b) is really needed to compute the power spectrum of a fully developed turbulent flow. Note that assumption (b) is the only experimental fact entering both the Kolmogorov theory and the estimate of its corrections.

It is clear that if we remove assumption (b) we cannot obtain the Kolmogorov spectrum, using the standard theoretical arguments given in the literature. We will now show that the physical model underlying our approach I is enough to reproduce the Kolmogorov spectrum in first approximation without any particular assumption on the energy dissipation. Obviously, we shall obtain at the end of our computation that energy dissipation is Reynolds number independent.

One of the most important physical quantities of a turbulent flow is the gradient of the velocity field. One usually thinks of a turbulent flow as having many scales of motion l_n, $l_0 = L$ being the external scale of energy input of the flow and $l_n = \lambda^{-n}L$, where λ is a number greater than one. For any given scale we assume that the dynamics can be described in terms of two different configurations (Tennekes 1968, Novikov and Stewart 1964, Frisch *et al* 1978, and I):

(1) 'laminar configurations', characterised by *low* values of the gradient of the velocity field;

(2) 'intermittent configurations', characterised by *high* values of the gradient of the velocity field.

In other words, we shall say that when a flow at scale l_n has a velocity gradient equal to V/L (where V is the typical large-scale velocity) it is in a laminar configuration, while when it has a velocity gradient equal to V_n^{i}/l_n (where V_n^{i} is the characteristic intermittent velocity at scale l_n) we shall use the expression intermittent configurations. We speak of two different configurations because the transport properties of a turbulent flow are strongly linked to the gradient of its velocity field, or in other words to its vorticity.

Let P_n be the probability that the flow is in an intermittent configuration. By definition, the average value at scale l_n of any increasing function $f(\cdot)$ of the gradient of the velocity field is given by

$$\left\langle f\left(\frac{\partial V}{\partial x}\right)_n\right\rangle \simeq P_n f\left(\frac{V_n^{\mathrm{i}}}{l_n}\right) + (1-P_n) f\left(\frac{V}{L}\right) \simeq P_n f\left(\frac{V_n^{\mathrm{i}}}{l_n}\right) \tag{2}$$

because $V_n^{\mathrm{i}}/l_n \gg V/L$ (Frisch *et al* 1978).

Our first step is to find model equations for V_n^{i} and P_n. We concentrate our attention first on V_n^{i}.

As long as the effects of viscosity on the dynamics of the scale l_n are very weak, there is no energy dissipation into heat. It is then reasonable to assume that in a first approximation all the characteristic intermittent velocities are the same, i.e.

$$V_n^{\mathrm{i}} = V \tag{3}$$

(an assumption like this was previously used by Saffman 1968). Therefore the average velocity gradient at the scale l_n is of order $(P_n V^2)^{1/2}/l_n = VP_n^{1/2}/l_n$. The quantity $V_n^{\mathrm{t}} (P_n V^2)^{1/2}$ will be called the typical velocity at scale l_n. The difference between the typical velocity and the characteristic velocity for a given scale is the following: the characteristic velocity refers to the instantaneous input of energy into the scale l_n, while the typical velocity refers to the average input of energy.

To compute P_n we follow the idea used previously in I. We think of the gradient of the velocity field δV_n as satisfying the following system of stochastic differential equations:

$$\mathrm{d}\delta V_n = F_n[\{\delta V_n\}]\,\mathrm{d}t + \sigma_n\,\mathrm{d}W(t) \tag{4}$$

where $\mathrm{d}W(t)$ is a Wiener process.

It is possible to show (Ventzel and Freidlin 1970) that the stochastic process given by equation (4) is equivalent to a Markov chain whose states are the stationary points (fixed points and limit cycles) of the deterministic equation

$$\mathrm{d}\delta V_n/\mathrm{d}t = F_n[\{\delta V_n\}].$$

It follows that the stochastic process δV_n jumps at random times between the stable (laminar) and unstable (intermittent) stationary states of the system (for more detailed explanation see I).

Using ergodic arguments it is possible to show that the probability of being in an intermittent configuration at scale l_n is equal to the average time τ_n the flow spends in an intermittent configuration divided by the average time T of a laminar configuration:

$$P_n = \tau_n / T. \tag{5}$$

To compute τ_n and T we can use the following approach. For any given scale the characteristic time of the evolution of the flow dynamics is strongly connected to the gradient of its velocity field. Therefore we can estimate τ_n and T to be given respectively by the expressions

$$\tau_n = (VP_n^{1/2}/l_n)^{-1}, \qquad T = (V/L)^{-1}. \tag{6}$$

Note that all the estimates given by energy arguments agree with this intuitive idea (Joseph 1976). Inserting (6) into (5) we obtain (in the inertial range)

$$P_n = (l_n/L)^{2/3}. \tag{7}$$

We can now use equation (7) to estimate the energy dissipation of the flow. By its very definition $\langle \epsilon \rangle$ is given by

$$\langle \epsilon \rangle = \nu \sum_{n=0}^{N} \frac{P_n V^2}{l_n^2}. \tag{8}$$

The value of N refers to the scale l_N at which the local Reynolds number is of order one. For any given scale the local Reynolds number R_n can be computed using the typical velocity previously defined:

$$R_n = \frac{V_n^{\mathrm{t}} l_n}{\nu} = \frac{V_n l_n}{\nu}\left(\frac{l_n}{L}\right)^{1/3} = \frac{VL}{\nu}\left(\frac{l_n}{L}\right)^{4/3} = R\left(\frac{l_n}{L}\right)^{4/3} \tag{9}$$

where R is the Reynolds number of the whole flow. From equation (9) we obtain

$$R_N = (l_N/L)^{4/3} R \simeq \mathrm{O}(1)$$

or in other words

$$l_N = LR^{-3/4} \tag{10}$$

i.e.

$$N = \tfrac{3}{4}(\ln R)/(\ln \lambda). \tag{11}$$

Using equations (10) and (11) in equation (8) we easily obtain

$$\langle \epsilon \rangle = \nu V^2 \sum_{n=1}^{N} \frac{P_n}{l_n^2} \simeq \nu V^2 \left(\frac{l_N}{L}\right)^{2/3} \frac{1}{l_N^2}$$

$$= \nu \frac{V^2}{L^2}\left(\frac{l_N}{L}\right)^{-4/3} = R^{-1}\frac{V^3}{L}R = \frac{V^3}{L}. \tag{12}$$

Equation (12) shows that $\langle \epsilon \rangle$ is a Reynolds independent quantity expressed in terms of the large-scale structure of the flow.

We can now find the Kolmogorov law from the preceding results (equations (2), (3) and (7)).

We can define the average kinetic energy $E(K_n)$ at scale l_n:

$$\int_{K_n}^{K_{n+1}} E(K)\,\mathrm{d}K = P_n V_n^{\mathrm{i}2}. \tag{13}$$

Then

$$\sum_n P_n V_n^{\mathrm{i}2} = \int_0^\infty E(K)\,\mathrm{d}K.$$

From equation (13) it is possible to write the following equation for $E(K_n)$:

$$E(K_n) \simeq P_n V_n^{\mathrm{i}2} K_n^{-1}. \tag{14}$$

From equation (7) we easily reach our result using equation (14):

$$E(K_n) \propto (l_n/L)^{2/3} V^2 K_n^{-1} = (V^2/L^{2/3}) K_n^{-5/3} \tag{15}$$

in the inertial range.

Equations (15) and (12) show that it is possible to obtain the Kolmogorov spectrum without any assumption on the average energy dissipation of the flow.

Using our results, we can also estimate the Kurtosis F of the gradient of the velocity field defined as

$$F \equiv \langle(\partial V/\partial x)^4\rangle / \langle(\partial V/\partial x)^2\rangle^2.$$

Because the largest contributions to the velocity gradient are in the dissipative scale l_n, we obtain from equations (10) and (2)

$$F \simeq \frac{P_N V_N^{\mathrm{i}4}/l_N^4}{(P_N V_N^{\mathrm{i}2}/l_N^2)^2} \simeq R^{1/2}. \tag{16}$$

The experimental value of F is (Van Atta and Chen 1970)

$$F \sim R^\alpha, \qquad \alpha \simeq 0{\cdot}6,$$

in agreement with our result.

We now extend our previous result to the two-dimensional case. As is well known for a flow in two dimensions, the entropy is an integral of motion. This means that our previous assumption equation (3) cannot be true in this case even in a first approximation. We can however estimate V_n^{i}, just assuming that entropy is not dissipated from one scale to another as long as viscous effects are small. If this is the case, we can immediately write

$$V_n^{\mathrm{i}2}/l_n^2 = V^2/L^2$$

i.e.

$$V_n^{\mathrm{i}} = (l_n/L)V. \tag{17}$$

Substituting equation (17) into the formulation of P_n, we obtain

$$P_n = \frac{l_n}{V_n^{\mathrm{i}} P_n^{1/2}} \frac{V}{L} = \frac{L}{V P_n^{1/2}} \frac{V}{L}.$$

This means that P_n is a constant for all the scale. From this result we easily obtain

$$E(K_n) \propto P_n V_n^{\mathrm{i}2} K_n^{-1} \propto K_n^{-3}. \tag{18}$$

Equation (18) has been derived in a number of different ways by many authors (Batchelor 1969, Kraichnan 1967).

In this Letter, following simple ideas on the dynamics of a turbulent flow, we obtain the Kolmogorov spectrum for the three- and two-dimensional turbulence. Our result does not need any particular assumption on energy dissipation. Indeed, one of our theoretical results is that energy dissipation is estimated to be a Reynolds number independent quantity.

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